EQUATION FOR NONSTEADY-STATE COMBUSTION VELOCITY OF A POWDER

B. V. Novozhilov

An integral equation is obtained for the nonsteady-state combustion velocity of a powder. It is shown that the effect of a variable tangential stream of gases on the rate of burning (nonsteady-state erosion) can be calculated in a similar way as for the change of pressure. The solution of the equation in linear approximation is considered (rate of burning differs slightly from steady-state).

The majority of papers on the theory of nonsteady-state combustion of a powder are based on the idea, first expressed by Ya. B. Zel'dovich [1], concerning the principal role of the inertness of a preheated layer of the condensed phase (the inertness of all processes, with the exception of thermal conductivity in the solid powder, can be neglected with good accuracy). It is shown in this approximation [2] that nonsteady-state processes during the combustion of powders can be calculated by solving the heat conductivity equation in the condensed phase for a given initial temperature distribution and known relations between the combustion velocity and the surface temperature on the one hand and the pressure and temperature gradient at the surface on the other hand. These relations are obtained by scaling the steady-state dependence of the combustion velocity and surface temperature on the pressure and initial temperature of the powder. In addition, either an explicit pressure-time dependence or the pressure equation must be given.

Almost all papers on the theory of nonsteady-state combustion are devoted to investigating the effect of changing pressure on the combustion velocity of the powder. However, other factors must be taken into account similarly which affect the combustion velocity through the gas phase. The most important of these is the flow rate of gases tangential to the combustion surface. It is well known (see, for example, [3]) that the flow of gases over the surface of the powder can alter its rate of combustion. This phenomenon is known in the literature under the name of blowing or erosive combustion. It is obvious that in the case of a time-variable flow, the thermal inertness of the solid phase depends on the finite retardation of the combustion velocity relative to the magnitude of the flow at the instant being considered.

1. Basic Relations of the Theory of Nonsteady-State Combustion

We shall assume the well-known steady-state laws of erosion, i.e., the dependence of the combustion velocity of a fuel u° and its surface temperature T_1° on the initial temperature T_0 and the rate of erosive flow G°:

$$u^\circ = u^\circ(T_0, G^\circ), \qquad T_1^\circ = T_1^\circ(T_0, G^\circ) \tag{1.1}$$

By means of the expression for the temperature gradient near the surface of the powder under steady-state conditions

$$f^\circ = \frac{u^\circ}{\varkappa}(T_1^\circ - T_0) \tag{1.2}$$

where $\varkappa$ is the temperature conductivity of the solid phase, the steady-state relations (1.1) can be converted to the dependence of the velocity of combustion and surface temperature on the gradient and rate of erosive flow:

Moscow. Translated from Zhurnal Prikladnoi Mekhaniki i Teknicheskoi Fiziki, No. 4, pp. 73-78, July-August, 1970. Original article submitted March 9, 1970.

$$u^\circ = u^\circ_\cdot (f^\circ,\ G^\circ), \qquad T_1^\circ = T_1^\circ (f^\circ,\ G^\circ) \tag{1.3}$$

In the nonsteady-state case, when the flow rate is changing, the latter expressions remain valid as Eq. (1.3) represents the relation between quantities which refer to the inertialess region (the surface of the powder and the gas phase are inertialess). Therefore, the superscripts (degree symbols) defining the stationarity can be discarded.

Similarly, the other factors which affect the combustion velocity by means of the inertialess combustion zones can be taken into account (the flow of radiation absorbed wholly in the gas phase at the surface of the fuel serves as an example).

Before setting down the system of equations for the theory of nonsteady-state combustion, we introduce dimensionless variables. If u° is a certain value of the combustion velocity (for example, initial or average) corresponding to the pressure p° and flow rate G° under steady-state conditions, then the dimensionless combustion velocity, coordinates, and time are written in the form

$$v = \frac{u}{u^\circ}, \qquad \xi = \frac{u^\circ}{\varkappa} x, \qquad \tau = \frac{(u^\circ)^2}{\varkappa} t \tag{1.4}$$

where x and t are the normal coordinates and time. The temperature inside the powder and also the gradient and temperature at the surface are represented conveniently in the form

$$\theta = \frac{T - T_0}{T_1^\circ - T_0}, \qquad \varphi = \frac{f}{f^\circ}, \qquad \vartheta = \frac{T_1 - T_0}{T_1^\circ - T_0} \tag{1.5}$$

where T_1° is the surface temperature corresponding to the velocity u°, pressure p°, and flow rate G°. Finally, the dimensionless pressure and rate of tangential flow can be introduced as

$$\eta = p / p^\circ, \qquad g = G / G^\circ \tag{1.6}$$

In these variables the problem of the theory of nonsteady-state combustion of a powder is formulated in the following way: to find the combustion velocity $v(\tau)$ from the thermal-conductivity equation, taking account of the thermal inertness of the condensed phase

$$\frac{\partial \theta}{\partial \tau} = \frac{\partial^2 \theta}{\partial \xi^2} - v \frac{\partial \theta}{\partial \xi} \qquad (\xi < 0) \tag{1.7}$$

with initial and boundary conditions

$$\theta (\xi,\ 0) = \theta_0 (\xi), \qquad \theta (-\infty,\ \tau) = 0, \quad \theta (0,\ \tau) = \vartheta \tag{1.8}$$

for the conditions that the relations

$$v = v (\varphi,\ \eta,\ g), \qquad \vartheta = \vartheta (\varphi,\ \eta,\ g) \tag{1.9}$$

and also the dependence of the pressure and erosive flow on the time

$$\eta = \eta (\tau), \qquad g = g (\tau) \tag{1.10}$$

are specified.

If the process of nonsteady-state combustion takes place in a chamber, then instead of the latter expressions differential equations must be written which satisfy the functions $\eta(\tau)$ and $g(\tau)$ and also the corresponding initial conditions. In the case of variability of these functions throughout the volume of the chamber, the coordinates of a point of the combustion surface also enter into the problem.

In solving a problem in this setting, together with the combustion velocity the nonsteady-state temperature distribution in the powder thickness $\theta(\xi, \tau)$ must be found. This function is a by-product of the theory, as it is not essential for solving problems of internal ballistics (excepting certain special problems). The basic problem of the theory of nonsteady-state combustion consists in predicting the behavior of the combustion velocity $v(\tau)$ for the given relations $\eta(\tau)$ and $g(\tau)$ (or one of them). We shall transform the theory into a form in which there is no extraneous function of the two variables (from the point of view of internal ballistics).

2. Integral Equations for Nonsteady-State Combustion Velocity

Assuming that $\theta = 0$ is to the right of the powder surface ($\xi > 0$), we apply a Fourier transform to the thermal-conductivity equation (1.7):

$$F(k, \tau) = \int_{-\infty}^{0} \theta(\xi, \tau) e^{-ik\xi} d\xi \tag{2.1}$$

Taking account of the boundary conditions, the equation for the transformed function will have the form

$$\frac{dF}{d\tau} + (k^2 + ikv) F = \varphi - v\vartheta + ik\vartheta \tag{2.2}$$

with the initial conditions

$$F(k, 0) = \int_{-\infty}^{0} \theta_0(\xi) e^{-ik\xi} d\xi \tag{2.3}$$

The linear equation (2.2) has the solution

$$F(k, \tau) = \int_0^{\tau} [\varphi(\tau') - v(\tau')\vartheta(\tau') + ik\vartheta(\tau')] \exp[-k^2(\tau - \tau') - ikI] d\tau + F(k, 0) \exp(-k^2\tau - ikJ) \tag{2.4}$$

where

$$I = \int_{\tau'}^{\tau} v(\tau'') d\tau'', \qquad J = \int_0^{\tau} v(\tau'') d\tau'' \tag{2.5}$$

Applying the inverse transform to Eq. (2.4):

$$\theta(\xi, \tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(k, \tau) e^{ik\xi} dk \tag{2.6}$$

we obtain

$$\theta(\xi, \tau) = \frac{1}{2\sqrt{\pi}} \left\{ \int_0^{\tau} \left(\varphi - v\vartheta + \frac{\vartheta(I - \xi)}{2(\tau - \tau')} \right) \exp \frac{-(I - \xi)^2}{2(\tau - \tau')} \frac{d\tau'}{\sqrt{\tau - \tau'}} + \frac{1}{\sqrt{\tau}} \int_{-\infty}^{0} \theta_0(u) \exp \frac{-(u + J - \xi)^2}{4\tau} du \right\} \tag{2.7}$$

There are three unknown functions of time in this expression – the velocity, gradient, and temperature at the surface. Two relations (1.9) between them are sufficient to determine them and to find $\theta(\xi, \tau)$. However, the relations between v, ϑ, and φ can be obtained if Eq. (2.7) is used at the point $\xi = 0$, i.e., at the powder surface. For this it must be remembered that when $\xi = 0$, the temperature undergoes a discontinuity (to the left it is equal to ϑ, and to the right it is zero); substituting $\xi = 0$ in Eq. (2.7), the root mean-square of the equation at the same time must be multiplied by two. We then have

$$\vartheta(\tau) = \frac{1}{\sqrt{\pi}} \left\{ \int_0^{\tau} \left(\varphi - v\vartheta + \frac{\vartheta I}{2(\tau - \tau')} \right) \exp \frac{-I^2}{4(\tau - \tau')} \frac{d\tau'}{\sqrt{\tau - \tau'}} + \frac{1}{\sqrt{\tau}} \int_{-\infty}^{0} \theta_0(u) \exp \frac{-(u + J)^2}{4\tau} du \right\} \tag{2.8}$$

Taking account also of the transient relations

$$v = v(\varphi, \eta, g), \qquad \vartheta = \vartheta(\varphi, \eta, g) \tag{2.9}$$

we have a closed system for determining any of the functions v, ϑ, or φ with respect to the given relations $\eta(\tau)$ and $g(\tau)$.

If necessary, the temperature distribution in the powder at any instant of time also can be found from Eq. (2.7).

The most interesting quantity is the combustion velocity. If the explicit form of the functions (2.9) is known, then it will be possible always to represent Eqs. (2.8) and (2.9) in the form of a single integral equation for $v(\tau)$, the value of which at a given time will depend on the entire history of change of the external parameters $\eta(\tau)$ and $g(\tau)$. The equation is nonlinear, as in the first place the thermal conductivity in the starting equation is a nonlinear term, corresponding to the convective flow of heat and, secondly the relations in Eqs. (2.9) in the general case are nonlinear.

In solving problems of internal ballistics, Eq. (2.8) is preferable to the original system of Eqs. (1.7)-(1.10) for the following reasons. First of all, there is a need for finding the appropriate function of the two variables $\theta(\xi, \tau)$. Obviously, this reduces to a considerable simplification of the numerical solution of problems which have no analytical solution. Further, a number of problems of nonsteady-state combustion theory can be solved by expansion in series with respect to a small parameter, say, with respect to the amplitude of the harmonically varying pressure. In this case, the use of the integral equation leads to a considerable simplification of the calculations – the calculations will not involve a different kind of correction to the steady-state temperature distribution. Finally, the meaning of the equation obtained consists in that it closes the system of equations of internal ballistics in which the pressure and combustion velocity are included in other quantities. When they are constant or are changing slowly (quasisteady-state conditions), the system of internal ballistics equations is closed by the steady-state relation $u° = (p°, T_0)$. In the nonsteady-state case this relation must be replaced by the integral relation (2.8) with the supplementary conditions of Eqs. (2.9). Of course, the model of the theory in the form of Eqs. (1.7)-(1.10) can be used also for the same purpose; however, in this case the system of internal ballistics equations is considerably more complicated, as the additional function of two variables is involved – the temperature within the bulk of the powder.

3. Linear Approximation, Steady-State Conditions of Fluctuating Combustion Velocity

Let us suppose that the pressure is varying according to the harmonic law

$$\eta = 1 + \eta_1 \cos \omega\tau, \quad \eta_1 \ll 1$$

We shall find in linear approximation the steady-state conditions for the combustion velocity. This condition corresponds to $\tau \to \infty$, when the effect of the initial conditions disappears. The term of the integral equation which contains the initial temperature distribution vanishes because of the factor $1/\sqrt{\tau}$. In linear approximation the method of complex amplitudes can be used, i.e., it is assumed that

$$\eta = 1 + \eta_1 e^{i\omega\tau}, \qquad v = 1 + v_1 e^{i\omega\tau}$$

$$\varphi = 1 + \varphi_1 e^{i\omega\tau}, \qquad \vartheta = 1 + \vartheta_1 e^{i\omega\tau}$$

The following relations are obtained from Eqs. (1.9) when g = const between small corrections:

$$v_1 = \frac{k}{k+r-1}\varphi_1 + \frac{\delta - \nu}{k+r-1}\eta_1, \quad \vartheta_1 = \frac{r}{k+r-1}\varphi_1 - \frac{\delta+\mu}{k+r-1}\eta_1 \tag{3.1}$$

$$k = (T_1° - T_0)\left(\frac{\partial \ln u°}{\partial T_0}\right)_p, \quad r = \left(\frac{\partial T_1°}{\partial T_0}\right)_p, \quad \nu = \left(\frac{\partial \ln u°}{\partial \ln p}\right)_{T_0} \tag{3.2}$$

$$\mu = \frac{1}{(T_1° - T_0)}\left(\frac{\partial T_1°}{\partial \ln p}\right)_{T_0}, \quad \delta = \frac{\partial (u°, T_1°)}{\partial (p, T_0)} = \nu r - \mu k$$

The integral I which occurs in the equation has the form

$$I = \tau - \tau' + I_1, \quad I_1 = \frac{v_1}{i\omega}(e^{i\omega\tau} - e^{i\omega\tau'})$$

If we substitute this expression in Eq. (2.8) and retain only terms of zero or first order, we obtain

$$1 + \vartheta_1 e^{i\omega\tau} = \frac{1}{\sqrt{\pi}}\int_0^\tau \left[\frac{1}{2} + \left(\varphi_1 - v_1 - \frac{\vartheta_1}{2} + \frac{v_1}{4i\omega}\right)e^{i\omega\tau'} - \frac{v_1}{4i\omega}e^{i\omega\tau} + \frac{v_1(e^{i\omega\tau} - e^{i\omega\tau'})}{2i\omega(\tau - \tau')}\right]\exp\frac{\tau' - \tau}{4}\frac{d\tau'}{\sqrt{\tau - \tau'}}$$

The integrals which occur in this expression must be taken for the condition $\tau \to \infty$. After integration we have

$$\vartheta_1 = \left(\varphi_1 - v_1 - \frac{v_1}{2} + \frac{v_1}{4i\omega}\right)\frac{2}{2z+1} - \frac{v_1}{2i\omega} + z\frac{v_1}{i\omega} \tag{3.3}$$

$$z = -{}^1/_2 + \sqrt{i\omega + {}^1/_4}$$

We obtainfrom Eqs. (3.1) and (3.3) the final relation between the amplitudes of the combustion velocity and the pressure [4]:

$$v_1 = \frac{\nu + \delta z}{1 - k + z(r + k / i\omega)} \eta_1 \tag{3.4}$$

The amplitude of the combustion velocity in the case of a harmonically varying tangential stream of gas can also be obtained quite similarly. Usually, the effect of the flow velocity on the combustion velocity starts to be expressed after a certain threshold value of the flow G_k. We shall suppose that the flow G changes harmonically but in such a way that its minimum value exceeds the threshold value. Then,

$$g = 1 + g_1 e^{i\omega\tau}$$

and the expressions for v, φ, and ϑ will have the previous form so that their unique values will correspond to the steady-state cycle for a flow value of G°.

It is obvious that calculations similar to those above will give the relation between v_1 and g_1, similar to Eq. (3.4):

$$v_1 = \frac{\nu' + \delta' z}{1 - k + z(r' + k' / i\omega)} g_1 \tag{3.5}$$

where the primed quantities are related with the derivatives of the combustion velocity and the surface temperature by the initial temperature and the tangential flow at the point corresponding to the steady-state cycle:

$$k' = (T_1^\circ - T_0)\left(\frac{\partial \ln u^\circ}{\partial T_0}\right)_G, \quad r' = \left(\frac{\partial T_1^\circ}{\partial T_0}\right)_G, \quad \nu' = \left(\frac{\partial \ln u^\circ}{\partial \ln G}\right)_{T_0}$$
$$\mu' = \frac{1}{T_1^\circ - T_0}\left(\frac{\partial T_1^\circ}{\partial \ln G}\right)_{T_0}, \quad \delta' = \frac{\partial (u^\circ, T_1^\circ)}{\partial (G, T_0)} = \nu' r' - \mu' k' \tag{3.6}$$

The quantities k' and ν' usually are measured in experiments to investigate erosive combustion under steady-state conditions. In order to solve the problem of nonsteady-state blowing, data are also necessary concerning the dependence of the surface temperature on the initial temperature and the tangential flow.

4. Linear Approximation, Transient Process

Under steady-state conditions the integral equation, in essence, need not be solved as the nature of the dependence of the combustion velocity on time is known. We shall proceed now to the case of solving the integral equation. We shall consider the combustion process with varying pressure in linear approximation:

$$\eta = 1 + \eta_1(\tau), \qquad \eta_1 \ll 1$$

and we shall take the initial temperature distribution in the form

$$\theta_0(\xi) = e^{\xi}$$

which corresponds to steady-state conditions when p= 1. The combustion velocity, gradient, and temperature at the boundary differ little from unity, i.e.,

$$v = 1 + v_1, \quad \varphi = 1 + \varphi_1, \quad \vartheta = 1 + \vartheta_1, \quad v_1 \sim \varphi_1 \sim \vartheta_1 \sim \eta_1 \ll 1$$

The integral equation (2.8) in linear approximation assumes the form

$$1 + \vartheta_1 = \frac{1}{\sqrt{\pi}} \int_0^\tau \left[\frac{1}{2} + \varphi_1 - v_1 - \frac{\vartheta_1}{2} + \frac{I_1}{2}\left(\frac{1}{\tau - \tau'} - \frac{1}{2}\right)\right] \exp\frac{\tau' - \tau}{4} \frac{d\tau'}{\sqrt{\tau - \tau'}}$$
$$+ \frac{1}{\sqrt{\pi\tau}} \int_0^\infty \left[1 - \frac{J_1(\tau - u)}{2\tau}\right] \exp\frac{-(u + \tau)^2}{4\tau} du \tag{4.1}$$
$$I_1 = \int_{\tau'}^{\tau} v_1(\tau'')\, d\tau'', \qquad J_1 = \int_0^\tau v_1(\tau'')\, d\tau''$$

After integration (in the term containing I_1, we change the order of integration), we obtain

$$\vartheta_1 = \frac{1}{\sqrt{\pi}}\int_0^\tau \left(\varphi_1 - \frac{\vartheta_1}{2}\right)\exp\frac{\tau'-\tau}{4}\frac{d\tau'}{\sqrt{\tau-\tau'}} - \int_0^\tau \left[1 - \operatorname{erf}\left(\frac{\sqrt{\tau-\tau'}}{2}\right)\right] v_1 d\tau' \tag{4.2}$$

If we substitute the expressions for ϑ_1 and φ_1 by v_1 and η_1 from Eqs. (3.1), then we obtain the second-order Volterra integral equation

$$v_1(\tau) = \frac{\delta}{r}\eta_1(\tau) + \frac{1}{\sqrt{\pi}}\int_0^\tau \frac{e^{-1/4 u}}{\sqrt{u}}\left[\frac{2k+r-2}{2r}v_1(\tau-u)\right.$$
$$\left. - \frac{\delta-2\nu}{2r}\eta_1(\tau-u)\right]du + \frac{k}{r}\int_0^\tau \left(1 - \operatorname{erf}\frac{\sqrt{u}}{2}\right)v_1(\tau-u)\,du \tag{4.3}$$

Solving the equation in the normal way, by a Laplace transformation, we obtain the relations between the transforms of the velocity $v_1(p)$ and the pressure $\eta_1(p)$ (here p is the Laplace variable):

$$v_1(p) = \frac{\nu + \delta z(p)}{1 - k + (r + k/p)z(p)}\eta_1(p), \quad z(p) = -{}^1/_2 + \sqrt{p + {}^1/_4} \tag{4.4}$$

This result was obtained earlier [5] by solving system (1.7)-(1.10).

LITERATURE CITED

1. Ya. B. Zel'dovich, "The theory of combustion of powders and explosives," Zh. Éksp. i Teor. Fiz., 12, No. 11, 12 (1942).
2. B. V. Novozhilov, "Theory of nonsteady-state combustion of homogeneous powders," Fizika Goreniya i Vzryva, No. 4 (1968).
3. R. N. Wimpress, Internal Ballistics of Solid-Fuel Rockets, McGraw-Hill (1950).
4. B. V. Novozhilov, "Combustion of a powder with harmonically changing pressure," Zh. Priklad. Mekh. i Tekh. Fiz., No. 6 (1965).
5. B. V. Novozhilov, "Nonsteady-state combustion of powders having a varying surface temperature," Zh. Priklad. Mekh. i Tekh. Fiz., No. 1 (1967).